CYCLIC CLASSES AND ATTRACTION CONES IN MAX ALGEBRA

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ABSTRACT. In max algebra it is well-known that the sequence A^k, with A an irreducible square matrix, becomes periodic at sufficiently large k. This raises a number of questions on the periodic regime of A^k and $A^k \otimes x$, for a given vector x. Also, this leads to the concept of attraction cones in max algebra, by which we mean sets of vectors with ultimate orbit period not exceeding a given number.

This paper shows that some of these questions can be solved by matrix squaring $(A, A^2, A^4, \ldots)$, analogously to recent findings of Semančíková [39, 40] concerning the orbit period in max-min algebra. Hence the computational complexity of such problems is $O(n^3 \log n)$. The main idea is to apply an appropriate diagonal similarity scaling $A \mapsto X^{-1} A X$, called visualization scaling, and to study the role of cyclic classes of the critical graph.

For powers of a visualized matrix in the periodic regime, we observe remarkable symmetry described by circulants and their rectangular generalizations. We exploit this symmetry to derive a system of equations for attraction cone, and present an algorithm which computes the coefficients of the system.

1. INTRODUCTION

By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max, \times}$ which is the set of nonnegative numbers $\mathbb{R}_+$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. Zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \sum_k^{\oplus} a_{ik} b_{kj} = \max_k(a_{ik} b_{kj})$ for all i, j. If A is a square matrix over $\mathbb{R}_+$ then the iterated product $A \otimes A \otimes \ldots \otimes A$ in which the symbol A appears k times will be denoted by A^k.

The *max-plus semiring* $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$, developed over the set of real numbers $\mathbb{R}$ with adjoined element $-\infty$ and the ordinary addition playing the role of multiplication, is another isomorphic "realization" of max algebra. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. In the max-plus setting, the zero element is $-\infty$ and the unity is 0.

The *min-plus semiring* $\mathbb{R}_{\min,+} = (\mathbb{R} \cup \{+\infty\}, \oplus = \min, \otimes = +)$ is also isomorphic to $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$. Another well-known semiring is the *max-min semiring* $\mathbb{R}_{\max,\min} = (\mathbb{R} \cup \{-\infty\} \cup \{\infty\}, \oplus = \max, \otimes = \min)$, see [25, 39, 40], but it is *not* isomorphic to any of the semirings above.

Max algebraic column spans of nonnegative matrices $A \in \mathbb{R}_+^{n \times n}$ are sets of max linear combinations of columns $\bigoplus_{i=1}^n \alpha_i A_{\cdot i}$ with nonnegative coefficients α_i. Such column spans are *max cones*, meaning that they are closed under componentwise maximum $\oplus$ and multiplication by nonnegative scalars. There are important analogies and links between max cones and convex cones [14, 17, 42, 41].

The maximum cycle geometric mean $\lambda(A)$, see below for exact definition, is one of the most important charasteristics of a matrix $A \in \mathbb{R}_+^{n \times n}$ in max algebra. In particular, it is the largest eigenvalue of the spectral problem $A \otimes x = \lambda x$. The cycles at which this maximum geometric mean is attained, are called *critical*. Further, one consideres the *critical graph* $\mathcal{C}(A)$ which consists of all nodes and edges that belong to the critical cycles. This graph is crucial for the description of eigenvectors [3, 15, 26].

The well-known *cyclicity theorem* states that if A is irreducible, then the sequence A^k becomes periodic after some finite transient time, and that the ultimate period of A^k is equal to the cyclicity of the critical graph [3, 15, 26]. Generalizations to reducible case, computational complexity issues and important special cases of this result have been extensively studied in [16, 24, 25, 32, 33].

In this paper we study the behaviour of matrix powers and orbits $A^k \otimes x$ in the irreducible case in the periodic regime, i.e., after the periodicity is reached. One of the main ideas is to study the periodicity of *visualized* matrices, meaning matrices with all entries less than or equal to the maximum cycle geometric mean. This study provides a connection to the theory of Boolean matrices [6, 29].

In Boolean matrix algebra, one considers components of imprimitivity of a matrix [6, 29], or equivalently, cyclic classes of the associated digraph [4]. In max algebra, cyclic classes of the critical graph have been considered as an important tool in the proof of the cyclicity theorem mentioned above, see [26] Sect. 3.1. Recently, the cyclic classes appeared in max-min algebra [39, 40], where they were used to study the ultimate periods of orbits and other periodicity problems. It was shown that such questions can be solved by matrix squaring $(A, A^2, A^4, A^8, \ldots)$, which yields computational complexity $O(n^3 \log n)$.

We show that the problems of computing ultimate period and matrix powers in the periodic regime can be solved by matrix squaring in *max algebra*, which yields the same complexity bound $O(n^3 \log n)$. This is achieved by exploiting visualization, and cyclic classes of the critical graph. Further it turns out that the periodic powers of visualized matrices have remarkable symmetry described by circulant matrices and their rectangular generalizations. We use this symmetry to derive a system of equations for *attraction cone*, meaning the max cone which consists of all vectors x whose ultimate period of $A^k \otimes x$ does not exceed a given number. We also describe extremals of attraction cones and present an algorithm for computing the coefficients of this system in the case when $\mathcal{C}(A)$ is strongly connected.

The contents of the paper are as follows. In Section 2 we revise two important topics in max algebra, namely the spectral problem and Kleene stars. In Section 3, we speak of the visualization and the connection to the theory of Boolean matrices which it provides, see Propositions 3.1 and 3.3. In Section 4, we study basic properties of matrix powers in the periodic regime, see Propositions 4.5 – 4.7. The problems which can be solved by matrix squaring are described in Theorem 4.11. In Section 5 we introduce some useful constructions associated with irreducible visualized matrices and their powers, namely, core matrix, CSR-representation, and describe their circulant simmetries in Proposition 5.4. In Section 6 we derive a concise system for attraction cone, see Theorem 6.2. We also describe extremals and present an algorithm for computing the coefficients of this system in the case when $\mathcal{C}(A)$ is strongly connected. We conclude with Section 7 which is devoted to numerical examples.

As $\mathbb{R}_{\max,+}$ and $\mathbb{R}_{\max,\times}$ are isomorphic, we use the possibility to switch between them, but only when it is really convenient. Thus, while the theoretical results are obtained over max-times semiring, which looks more natural in connection with diagonal matrix scaling and Boolean matrices, the examples in Section 7 are written over *max-plus semiring*, where it is much easier to calculate.

We remark that some aspects of the theory of attraction spaces have been investigated in [5, 18, 30] in certain special cases. Also, the periodicity of max algebraic powers of matrices can be regarded from the viewpoint of max-plus semigroups as studied in [31].

2. Two topics in max algebra

2.1. **Spectral problem.** Let $A \in \mathbb{R}_+^{n \times n}$. Consider the problem of finding $\lambda \in \mathbb{R}_+$ and nonzero $x \in \mathbb{R}_+^n$ such that

$$(1) \qquad\qquad\qquad A \otimes x = \lambda x.$$

If for some λ there exists a nonzero $x \in \mathbb{R}_+^n$ which satisfies (1), then λ is called a *max-algebraic eigenvalue* of A, and x is a *max-algebraic eigenvector* of A associated with λ. With the zero vector adjoined, the set of max-algebraic eigenvectors associated with λ forms a max cone, which is called the *eigencone* associated with λ.

The largest max-algebraic eigenvalue of $A \in \mathbb{R}_+^{n \times n}$ is equal to

$$(2) \qquad \lambda(A) = \bigoplus_{k=1}^{n} (\mathrm{Tr}_\oplus A^k)^{1/k},$$

where $\mathrm{Tr}_\oplus$ is defined by $\mathrm{Tr}_\oplus(A) := \bigoplus_{i=1}^{n} a_{ii}$ for any $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. Further we explain the graph-theoretic meaning of (2), assumed that $\lambda(A) \neq 0$.

With $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ we can associate the weighted digraph $\mathcal{D}(A) = (N(A), E(A))$, with the set of nodes $N(A) = \{1, \dots, n\}$ and the set of edges $E(A) = \{(i,j) \mid a_{ij} \neq 0\}$ with weights $w(i,j) = a_{ij}$. Suppose that $\pi = (i_1, ..., i_p)$ is a path in $\mathcal{D}(A)$, then the *weight* of π is defined to be $w(\pi, A) = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{p-1} i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then π is called a cycle. One can check that

$$\lambda(A) = \max_\sigma \mu(\sigma, A),$$

where the maximization is taken over all cycles in $\mathcal{D}(A)$ and

$$\mu(\sigma, A) = w(\sigma, A)^{1/k}$$

denotes the *geometric mean* of the cycle $\sigma = (i_1, ..., i_k, i_1)$. Thus $\lambda(A)$ is the *maximum cycle geometric mean* of $\mathcal{D}(A)$.

$A \in \mathbb{R}_+^{n \times n}$ is *irreducible* if for any nodes i and j there exists a path in $\mathcal{D}(A)$, which begins at i and ends at j. In this case A has a unique max-algebraic eigenvalue which equals $\lambda(A)$.

Note that $\lambda(\alpha A) = \alpha \lambda(A)$ and hence $\lambda(A/\lambda(A)) = 1$ if $\lambda(A) > 0$. Unless we need matrices with $\lambda(A) = 0$, we can always assume without loss of generality that $\lambda(A) = 1$. Such matrices will be called *definite*.

An important relaxation of (1) is

$$(3) \qquad\qquad\qquad A \otimes x \leq \lambda x.$$

The nonzero vectors $x \in \mathbb{R}_+^n$ which satisfy (3) are called *subeigenvectors* associated with λ. With the zero vector adjoined, they form a max cone called *subeigencone*. This is a conventionally convex cone, meaning that it is closed under the *ordinary* addition. See [42] for more details.

The eigencone (resp. subeigencone) of A associated with $\lambda(A)$ will be denoted by $V(A)$ (resp. $V^*(A)$).

2.2. **Kleene stars.** Let $A \in \mathbb{R}_+^{n \times n}$. Consider the formal series

(4)
$$A^* = I \oplus A \oplus A^2 \oplus \ldots,$$

where I denotes the identity matrix with entries

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Series (4) is a max-algebraic analogue of $(I - A)^{-1}$, and it converges to a matrix with finite entries if and only if $\lambda(A) \leq 1$ [3, 11]. In this case

(5)
$$A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1},$$

which is called the *Kleene star* of A.

For any $A \in \mathbb{R}_+^{n \times n}$,

(6)
$$A \text{ is a Kleene star } \Leftrightarrow A^2 = A, \ a_{ii} = 1 \ \forall i.$$

The condition $\lambda(A) \leq 1$ suggests that there is a strong interplay between Kleene stars and spectral problems. To describe this in more detail, we need the following notions and notation.

A cycle σ in $\mathcal{D}(A)$ is called *critical*, if $\mu(\sigma, A) = \lambda(A)$. Every node and edge that belongs to a critical cycle is called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$. The *critical digraph* of A, further denoted by $\mathcal{C}(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of $\mathcal{D}(A)$. For definite $A \in \mathbb{R}_+^{n \times n}$, it follows that $a_{ij}a_{ji}^* \leq 1$ [3]. Further,

(7)
$$(i, j) \in E_c(A) \Leftrightarrow a_{ij}a_{ji}^* = 1.$$

For definite $A \in \mathbb{R}_+^{n \times n}$, the relation between Kleene star, critical graph and spectral problems is briefly as follows [3, 15, 42]:

(8)
$$V^*(A) = \text{span}(A^*) = \left\{ \bigoplus_{i=1}^n \alpha_i A_{\cdot i}^*, \ \alpha_i \in \mathbb{R}_+ \right\},$$

(9)
$$V(A) = \left\{ \bigoplus_{i \in N_c(A)} \alpha_i A_{\cdot i}^*, \ \alpha_i \in \mathbb{R}_+ \right\},$$

(10)
$$x \in V^*(A), (i, j) \in E_c(A) \Rightarrow a_{ij}x_j = x_i.$$

Equation (8) means that $V^*(A)$ is the max-algebraic column span of Kleene star A^*, also called *Kleene cone*. This cone is convex in conventional sense. By (9), $V(A)$ is the max subcone of $V^*(A)$, spanned by the columns with critical indices. Implication (10) means that for any subeigenvector $x \in V^*(A)$ and $i \in N_c(A)$, the maximum in $\bigoplus_j a_{ij}x_j$

is attained at j such that $(i, j) \in E_c(A)$. In particular, $(A \otimes x)_i = x_i$ for all $x \in V^*(A)$ and $i \in N_c(A)$.

Not all columns in (8) and (9) are necessary. Let $\mathcal{C}(A)$ have $n_c \in \{1, \ldots, n\}$ strongly connected components (s.c.c.) $\mathcal{C}_\mu$, for $\mu = 1, \ldots, n_c$. It follows from the definition of $\mathcal{C}(A)$ that s.c.c. $\mathcal{C}_\mu$ are disjoint. The corresponding node sets will be denoted by N_μ. Let $\bar{c}$ denote the number of non-critical nodes of $\mathcal{D}(A)$. It can be shown [3, 15] that if i, j belong to the same s.c.c. of $\mathcal{C}(A)$, then the columns $A^*_{\cdot i}$ and $A^*_{\cdot j}$ are multiples of each other. The same holds for the rows $A^*_{i \cdot}$ and $A^*_{j \cdot}$. Hence

$$(11) \qquad V^*(A) = \left\{ \bigoplus_{i \in K} \alpha_i A^*_{\cdot i}, \ \alpha_i \in \mathbb{R}_+ \right\}$$

$$(12) \qquad V(A) = \left\{ \bigoplus_{i \in N_c(A) \cap K} \alpha_i A^*_{\cdot i}, \ \alpha_i \in \mathbb{R}_+ \right\},$$

where K is any set of indices which contains all non-critical indices and for every $\mathcal{C}_\mu$ there is a unique index of this component in K.

Consider A^*_{KK}, the principal submatrix of A^* extracted from the rows and columns with indices in K. Condition (6) implies that A^*_{KK} is itself a Kleene star. It follows from the maximality of $\mathcal{C}_\mu$ that there is a unique permutation of K that has the greatest weight with respect to A^*_{KK}. The *weight of a permutation* π of $\{1, \ldots, n\}$ with respect to $A \in \mathbb{R}^{n \times n}_+$ is defined as $\prod_{i=1}^n a_{i\pi(i)}$. Thus A^*_{KK} is *strongly regular* in the sense of Butkovič [7]. From this it can be deduced that the columns of A^* with indices in K are *independent*, meaning that none of them can be expressed as a max combination of the other columns. In other words [10], the columns of A^* with indices in K (resp., in $N_c(A) \cap K$) form a *basis* of $V^*(A)$ (resp., of $V(A)$). This basis is essentially unique [10], meaning that any other basis can be obtained from it by scalar multiplication.

More precisely, the strong regularity of A^*_{KK} is equivalent to saying that this basis is *tropically independent*, hence the tropical rank of A^* is equal to $n_c + \bar{c}$, see [2, 27, 28] for definitions and further details.

3. Visualization and Boolean matrices

3.1. **Visualization.** Consider a positive $x \in \mathbb{R}^n_+$ and define

$$(13) \qquad X = \mathrm{diag}(x) := \begin{pmatrix} x_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & x_n \end{pmatrix}$$

The transformation $A \mapsto X^{-1}AX$ is called a *diagonal similarity scaling* of A. Such transformations do not change $\lambda(A)$ and $\mathcal{C}(A)$ [21]. They commute with max-algebraic

multiplication of matrices and hence with the operation of taking the Kleene star. Geometrically, they correspond to automorphisms of $\mathbb{R}_+^n$, both in the case of max algebra and in the case of nonnegative linear algebra. Further we define scalings which lead to particularly convenient forms of matrices in max algebra.

A definite matrix $A \in \mathbb{R}_+^{n \times n}$ is called *visualized*, if

$$(14) \qquad\qquad a_{ij} \leq 1, \ \forall i, j = 1, \ldots, n$$

$$(15) \qquad\qquad a_{ij} = 1, \ \forall (i,j) \in E_c(A)$$

A visualized matrix $A \in \mathbb{R}_+^{n \times n}$ is called *strictly visualized* if

$$(16) \qquad\qquad a_{ij} = 1 \Leftrightarrow (i,j) \in E_c(A).$$

Visualization scalings were known already to Afriat [1] and Fiedler-Pták [23], and motivated extensive study of matrix scalings in nonnegative linear algebra, see e.g. [21, 22, 36, 38]. We remark that some constructions and facts related to application of visualization scaling in max algebra have been observed in connection with max algebraic power method [19, 20], behaviour of matrix powers [8] and max-balancing [36, 38].

Visualization scalings are described in [42] in terms of the subeigencone $V^*(A)$ and its relative interior. For the convenience of the reader, we show their existence for any definite $A \in \mathbb{R}_+^{n \times n}$. In the proposition stated below, the summation in part 2. is *conventional*.

Proposition 3.1. *Let $A \in \mathbb{R}_+^{n \times n}$ be definite and $X = \mathrm{diag}(x)$.*

1. *If $x = \bigoplus_{i=1}^n A_{\cdot i}^*$ then $X^{-1}AX$ is visualized.*
2. *If $x = \sum_{i=1}^n A_{\cdot i}^*$ then $X^{-1}AX$ is strictly visualized.*

Proof. 1. Observe that $x \in V^*(A)$ and x is positive. Then $a_{ij}x_j \leq x_i$ for all i,j implies $x_i^{-1}a_{ij}x_j \leq 1$, and by (10) $x_i^{-1}a_{ij}x_j = 1$ for all $(i,j) \in E_c(A)$.
2. Observe that x is positive, and that $x \in V^*(A)$ since $V^*(A)$ is convex. Hence $X^{-1}AX$ is visualized. It remains to check that $(i,j) \notin E_c(A)$ implies $a_{ij}x_j < x_i$. We need to find k such that $a_{ij}a_{jk}^* < a_{ik}^*$. But this is true for $k = i$, since $a_{ii}^* = 1$ and $a_{ij}a_{ji}^* < 1$ by (7). This completes the proof. $\square$

More precisely [42], $A \in \mathbb{R}_+^{n \times n}$ can be visualized by any positive vector in $V^*(A)$, and it can be strictly visualized by any vector in the relative interior of $V^*(A)$.

3.2. **Max algebra and Boolean matrices.** Max algebra is related to the algebra of Boolean matrices. The latter algebra is defined over the Boolean semiring $\mathcal{S}$ which is the set $\{0,1\}$ equipped with logical operations "OR" $a \oplus b := a \vee b$ and "AND" $a \otimes b := a \wedge b$. Clearly, Boolean matrices can be treated as objects of max algebra, as a very special but crucial case.

For a strongly connected graph, its cyclicity is defined as the g.c.d. of the lengths of all cycles (or equivalently, all simple cycles). If the cyclicity is 1 then the graph is called *primitive*, otherwise it is called *imprimitive*. We will not distinguish between cyclicity (or primitivity) of a Boolean matrix A and the associated digraph $\mathcal{D}(A)$. Further we recall an important result of Boolean matrix theory.

Proposition 3.2 (Brualdi and Ryser [6])**.** *Let $A \in \mathcal{S}^{n \times n}$ be irreducible, and let γ_A be the cyclicity of $\mathcal{D}(A)$ (which is strongly connected). Then for each $k \geq 1$, there exists a permutation matrix P such that $P^{-1}A^kP$ has r irreducible diagonal blocks, where $r = \gcd(k, \gamma_A)$, and all elements outside these blocks are zero. The cyclicity of all these blocks is γ_A/r.*

In max algebra, let $A \in \mathbb{R}_+^{n \times n}$. Define the Boolean matrix $A^{[C]} = (a_{ij}^{[C]})$ by

$$(17) \qquad a_{ij}^{[C]} = \begin{cases} 1, & (i,j) \in E_c(A) \\ 0, & (i,j) \notin E_c(A). \end{cases}$$

Let $A, B \in \mathbb{R}_+^{n \times n}$. Assume that $\mathcal{C}(A)$ has n_c s.c.c. $\mathcal{C}_\mu$ for $\mu = 1, \ldots, n_c$, with cyclicities γ_μ. Denote by $B_{\mu\nu}$ the block of B extracted from the rows with indices in N_μ and columns with indices in N_ν.

The following proposition can be seen as a corollary of Proposition 3.2. The idea of the proof given below is due to Hans Schneider. See also [26] Section 3.1 and [8] Theorem 2.3.

Proposition 3.3. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) \neq 0$.*

1. $\lambda(A^k) = \lambda^k(A)$.
2. $(A^{[C]})^k = (A^k)^{[C]}$.
3. *For each $k \geq 1$, there exists a permutation matrix P such that $(P^{-1}A^kP)_{\mu\mu}^{[C]}$, for each $\mu = 1, \ldots, n_c$, has $r_\mu := \gcd(k, \gamma_\mu)$ irreducible blocks and all elements outside these blocks are zero. The cyclicity of all blocks in $(P^{-1}A^kP)_{\mu\mu}^{[C]}$ is equal to γ_μ/r_μ.*

Proof. We can assume that A is definite. Further, the diagonal similarity scaling commutes with max algebraic matrix multiplication and changes neither $\lambda(A)$ nor $\mathcal{C}(A)$ [21], and by Proposition 3.1, part 2, there exists a strict visualization scaling. Hence we can assume that A is strictly visualized. In this case $A^{[C]} = A^{[1]}$, where $A^{[1]} = (a_{ij}^{[1]})$ is defined by

$$(18) \qquad a_{ij}^{[1]} = \begin{cases} 1, & a_{ij} = 1, \\ 0, & a_{ij} < 1. \end{cases}$$

It is easily seen that $(A^{[1]})^k = (A^k)^{[1]}$. As $A^{[1]} = A^{[C]}$, all entries of $A^{[1]}$ outside the blocks $A_{\mu\mu}^{[1]}$ are zero, which assures that $(A^{[1]})_{\mu\mu}^k = (A_{\mu\mu}^{[1]})^k$.

Proposition 3.2 implies that part 3. is true for $(A^{[1]})^k = (A^k)^{[1]}$. This implies that $P^{-1}(A^k)^{[1]}P$ has irreducible blocks and $\lambda(A^k) = 1$, which shows part 1. Also, $P^{-1}(A^k)^{[1]}P$ has block structure where all diagonal blocks are irreducible and all off-diagonal blocks are zero. This implies $(A^k)^{[C]} = (A^k)^{[1]}$, and parts 2. and 3. follow immediately. □

3.3. **Cyclic classes.** For a path P in a digraph $G = (N, E)$, where $N = \{1, \ldots, n\}$, denote by $l(P)$ the length of P, i.e., the number of edges traversed by P.

Proposition 3.4 (Brualdi-Ryser [6]). *Let $G = (N, E)$ be a strongly connected digraph with cyclicity γ_G. Then the lengths of any two paths connecting $i \in N$ to $j \in N$ (with i, j fixed) are congruent modulo γ_G.*

Proposition 3.4 implies that the following *equivalence relation* can be defined: $i \sim j$ if there exists a path P from i to j such that $l(P) \equiv 0 \pmod{\gamma_G}$. The equivalence classes of G with respect to this relation are called *cyclic classes* [4, 39, 40]. The cyclic class of i will be denoted by $[i]$.

Consider the following *access relations* between cyclic classes: $[i] \rightarrow_t [j]$ if there exists a path P from a node in $[i]$ to a node in $[j]$ such that $l(P) \equiv t \pmod{\gamma_G}$. In this case, a path P with $l(P) \equiv t \pmod{\gamma_G}$ exists between any node in $[i]$ and any node in $[j]$. Further, by Proposition 3.4 the length of any path between a node in $[i]$ and a node in $[j]$ is congruent to t, so the relation $[i] \rightarrow_t [j]$ is well-defined. Classes $[i]$ and $[j]$ will be called *adjacent* if $[i] \rightarrow_1 [j]$.

Cyclic classes can be computed in $O(|E|)$ time by Balcer-Veinott digraph condensation, where $|E|$ denotes the number of edges in G. At each step of this algorithm, we look for all edges which issue from a certain node i, and condense all end nodes of these edges into a single node. A precise description of this method can be found in [4, 6]. We give an example of its work, see Figures 1 and 2.
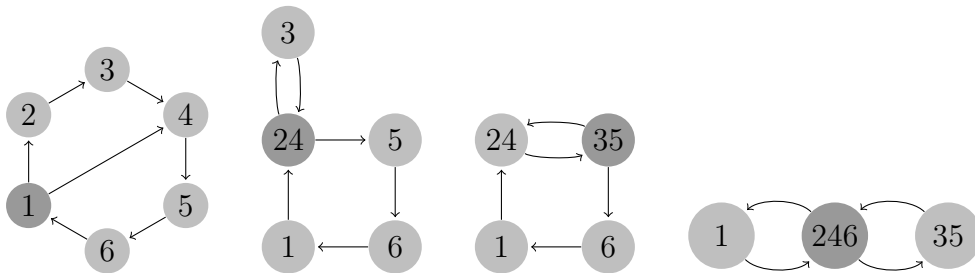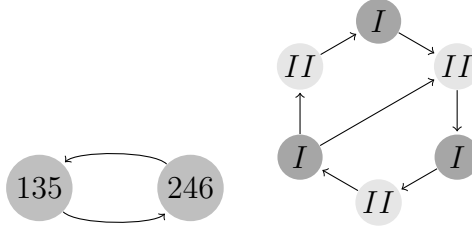

FIGURE 1. Balcer-Veinott algorithm

FIGURE 2. Result of the algorithm (left) and cyclic classes (right)

In this example, see Figure 1 at the left, we start by condensing nodes 2 and 4, which are "next to" node 1, into the node 24. Further we proceed with condensing nodes 3 and 5 into the node 35. In the end, see Figure 2 at the left, there are just two nodes 135 and 246. They correspond to two cyclic classes $\{1, 3, 5\}$ and $\{2, 4, 6\}$ of the initial graph, see Figure 2 at the right.

The notion of cyclic classes and access relations can be generalized to the case when G has n_c disjoint components G_μ with cyclicities γ_μ, for $\mu = 1, \ldots, n_c$ (just like the critical graph in max algebra). In this case we write $i \sim j$ if i, j belong to the same component and there exists a path P from i to j such that $l(P) \equiv 0 (\mathrm{mod}\, \gamma_\mu)$. If $l(P) \equiv t (\mathrm{mod}\, \gamma_\mu)$, then we write $[i] \rightarrow_t [j]$. In this case the *cyclicity* of G is $\gamma := \mathrm{lcm}\ \gamma_\mu$, $\mu = 1, \ldots, n_c$.

We will be interested in the cyclic classes of critical graphs, and below we also give an explanation of these, in terms of the Boolean matrix $A^{[C]}$. Let $A \in \mathbb{R}_+^{n \times n}$. Following Brualdi and Ryser [6] we can find such ordering of the indices that any submatrix $A_{\mu\mu}^{[C]}$, which corresponds to an imprimitive component $\mathcal{C}_\mu$ of $\mathcal{C}(A)$, will be of the form

$$(19) \quad \begin{pmatrix} \mathbf{0} & A_{s_1 s_2}^{[C]} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_{s_2 s_3}^{[C]} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{s_{k-1} s_k}^{[C]} \\ A_{s_k s_1}^{[C]} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix},$$

where k is the number of cyclic classes in $\mathcal{C}_\mu$. Indices s_i and s_{i+1} for $i = 1, \ldots, k-1$, and s_k and s_1 correspond to adjacent cyclic classes. By Proposition 3.3 part 2, when A is raised to power k, $A^{[C]}$ is also raised to the same power over the Boolean algebra. Any power of $A^{[C]}$ has a similar block-permutation form. In particular, $(A^{\gamma_\mu})_{\mu\mu}^{[C]}$ looks like

$$(20) \quad \begin{pmatrix} (A^{\gamma_\mu})_{s_1 s_1}^{[C]} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (A^{\gamma_\mu})_{s_2 s_2}^{[C]} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & (A^{\gamma_\mu})_{s_k s_k}^{[C]} \end{pmatrix}$$

Theorem 5.4.11 of [29] implies that the sequence $(A^k)^{[C]} = (A^{[C]})^k$ becomes periodic after $k \leq (n-1)^2 + 1$, with period $\gamma = \mathrm{lcm}(\gamma_\mu)$, $\mu = 1, \ldots, n_c$. In the periodic regime, all entries of nonzero blocks are equal to 1.

4. Periodicity and complexity

4.1. **Spectral projector and periodicity in max algebra.** We further assume that the critical graph $\mathcal{C}(A)$ occupies the first c nodes, i.e., that $N_c(A) = \{1, \ldots, c\}$.

For a definite $A \in \mathbb{R}_+^{n \times n}$, consider the matrix $Q(A)$ with entries

$$(21) \qquad q_{ij} = \bigoplus_{k=1}^{c} a_{ik}^* a_{kj}^*, \ i, j = 1, \ldots, n.$$

The max-linear operator whose matrix is $Q(A)$, is a max-linear *spectral projector* associated with A, in the sense that it projects $\mathbb{R}_+^n$ on the max-algebraic eigencone $\{x \mid A \otimes x = x\}$ [3]. We also note that $Q(A)$ is important for the policy iteration algorithm of [13].

We will need the following property of $Q(A)$ which follows directly from (21).

Proposition 4.1. *For a definite $A \in \mathbb{R}_+^{n \times n}$, any column (or row) of $Q(A)$ with index in $1, \ldots, c$ is equal to the corresponding column (or row) of A^*.*

This operator is closely related to the periodicity questions.

Theorem 4.2 (Baccelli et al. [3], Theorem 3.109)**.** *Let $A \in \mathbb{R}_+^{n \times n}$ be irreducible and definite, and let all s.c.c. of $\mathcal{C}(A)$ have cyclicity 1. Then there is an integer $T(A)$ such that $A^r = Q(A)$ for all $r \geq T(A)$.*

It can be easily shown that Theorem 4.2 can be generalized to the situation when A is in a blockdiagonal form with irreducible definite diagonal blocks $A_{11}, \ldots, A_{uu}$ so that

$$(22) \qquad A = \begin{pmatrix} A_{11} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & A_{uu} \end{pmatrix}$$

If A is irreducible and definite, then A^k for all $k \geq 1$ is in a blockdiagonal form (22), where all blocks $A_{11}, \ldots, A_{uu}$ are irreducible and definite.

Indeed, it is evident (when A is assumed to be visualised), that the m.c.g.m. of each block does not exceed 1. On the other hand, any column $A_{\cdot i}^*$ with $1 \leq i \leq c$ is a positive max-algebraic eigenvector of A and hence of A^γ. This shows that the m.c.g.m. of each submatrix, being equal to the largest max-algebraic eigenvalue of that submatrix, cannot be less than 1.

It follows from Proposition 3.3 part 3 that all components of $\mathcal{C}(A^\gamma)$ are primitive, where γ is the cyclicity of $\mathcal{C}(A)$.

These arguments lead us to the following extension of Theorem 4.2.

Theorem 4.3. *Let $A \in \mathbb{R}_+^{n \times n}$ be irreducible and definite, and let γ be the cyclicity of $\mathcal{C}(A)$. There exists $T(A)$ such that*

1. $A^{t\gamma} = Q(A^\gamma)$ *for all* $t\gamma \geq T(A)$;
2. $A^{r+\gamma} = A^r$ *for all* $r \geq T(A)$.

$T(A)$ will be called the *transient*, and the powers A^r for $r \geq T(A)$ will be called *periodic powers*.

It is also important that the entries $a_{ij}^{(r)}$, where i or j are critical, become periodic much faster than the non-critical part of A. The following proposition is a known result, which is proved here for convenience of the reader.

Proposition 4.4 (Nachtigall [34]). *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite irreducible matrix. Critical rows and columns of A^r become periodic for $r \geq n^2$.*

Proof. We prove the claim for rows, and for columns everything is analogous. Let $i \in \{1, \dots, c\}$. Then there is a critical cycle of length l to which i belongs. Hence $a_{ii}^{(kl)} = 1$ for $k \geq 1$. Since for all $m < k$ and any $t = 1, \dots, n$ we have

$$a_{is}^{(ml)} = a_{ii}^{((k-m)l)} a_{is}^{(ml)} \leq a_{is}^{(kl)},$$

it follows that

$$(23) \qquad\qquad a_{is}^{(kl)} = \bigoplus_{m=1}^{k} a_{is}^{(ml)}.$$

Entries $a_{is}^{(kl)}$ are maximal weights of paths of length k with respect to the matrix A^l. Since the weights of all cycles are less than or equal to 1 and all paths of length n are not simple, the maximum is achieved at $k \leq n$. Using (23) we obtain that $a_{is}^{((t+1)l)} = a_{is}^{(tl)}$ for all $t \geq n$. Further,

$$a_{is}^{(tl+d)} = \bigoplus_{k} a_{ik}^{(tl)} a_{ks}^{(d)},$$

and it follows that $a_{is}^{((t+1)l+d)} = a_{is}^{(tl+d)}$ for all $t \geq n$ and $0 \leq d \leq l-1$. Hence $a_{is}^{(k)}$ is periodic for $k \geq nl$, and all these sequences, for any $i = 1, \dots, c$ and any s, become periodic for $k \geq n^2$. $\square$

The number after which the critical rows and columns of A^t become periodic will be denoted by $T_c(A)$.

4.2. **The ultimate spans of matrices.** Max algebraic powers in the periodic regime have the following properties.

Proposition 4.5. *Let* $A \in \mathbb{R}_+^{n \times n}$ *be a definite and irreducible matrix, and let* $t \geq 0$ *be such that* $t\gamma \geq T(A)$. *Then for every integer* $l \geq 0$

$$(24) \qquad A_{k\cdot}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma+l}, \ A_{\cdot k}^{t\gamma+l} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma+l}, \quad 1 \leq k \leq n.$$

Proof. Due to Theorem 4.3, for $B = A^\gamma$ and any $r \geq T(B)$ we have

$$(25) \qquad b_{kj}^{(r)} = \bigoplus_{i=1}^{c} b_{ki}^* b_{ij}^*, \quad 1 \leq k, j \leq n.$$

By Theorem 4.3 and Proposition 4.1, we have $b_{ki}^* = b_{ki}^{(r)} = a_{ki}^{(t\gamma)}$ and $b_{ij}^* = b_{ij}^{(r)} = a_{ij}^{(t\gamma)}$ for all $r \geq T(B)$ or equivalently $t\gamma \geq T(A)$, and any $i \leq c$. Hence

$$(26) \qquad a_{kj}^{(t\gamma)} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} a_{ij}^{(t\gamma)}, \quad 1 \leq k, j \leq n.$$

In the matrix notation, this is equivalent to:

$$(27) \qquad A_{k\cdot}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ki}^{(t\gamma)} A_{i\cdot}^{t\gamma}, \ A_{\cdot k}^{t\gamma} = \bigoplus_{i=1}^{c} a_{ik}^{(t\gamma)} A_{\cdot i}^{t\gamma}, \quad 1 \leq k \leq n.$$

Multiplying (27) by any power A^l, we obtain (24). $\qquad \square$

In the proof of the next proposition we will use the following simple principle

$$(28) \qquad a_{ij}^{(r)} a_{jk}^{(s)} \leq a_{ik}^{(r+s)}, \quad \forall i, j, k, r, s,$$

which holds for the matrix powers in max algebra.

Proposition 4.6. *Let* $A \in \mathbb{R}_+^{n \times n}$ *be a definite and irreducible matrix, and let* $i, j \in \{1, \ldots, c\}$ *be such that* $[i] \to_l [j]$, *for some* $0 \leq l < \gamma$.

    1. *For any* $r \geq T_c(A)$, *there exists* $t_1 \geq 0$ *such that*

$$(29) \qquad a_{ij}^{(t_1\gamma+l)} A_{\cdot i}^r = A_{\cdot j}^{(r+l)}, \ a_{ij}^{(t_1\gamma+l)} A_{j\cdot}^r = A_{i\cdot}^{r+l}.$$

    2. *If* A *is visualized, then for all* $r \geq T_c(A)$

$$(30) \qquad A_{\cdot i}^r = A_{\cdot j}^{r+l}, \ A_{j\cdot}^r = A_{i\cdot}^{r+l}.$$

Proof. If $[i] \to_l [j]$ then $[j] \to_s [i]$ where $l + s = \gamma$. By the definition of access relations there exists a critical path of length $t_1\gamma + l$ connecting i to j, and a critical path of

length $t_2\gamma + s$ connecting j to i. Hence $a_{ij}^{(t_1\gamma+l)}a_{ji}^{(t_2\gamma+s)} = 1$, and in the visualized case $a_{ij}^{(t_1\gamma+l)} = a_{ji}^{(t_2\gamma+s)} = 1$. Combining this with (28) we obtain

$$
(31) \quad
\begin{aligned}
A_{\cdot i}^r &= A_{\cdot i}^r a_{ij}^{(t_1\gamma+l)}a_{ji}^{(t_2\gamma+s)} \leq A_{\cdot j}^{r+t_1\gamma+l}a_{ji}^{(t_2\gamma+s)} \leq A_{\cdot i}^{r+(t_1+t_2+1)\gamma}, \\
A_{j\cdot}^r &= A_{j\cdot}^r a_{ij}^{(t_1\gamma+l)}a_{ji}^{(t_2\gamma+s)} \leq A_{i\cdot}^{r+t_1\gamma+l}a_{ji}^{(t_2\gamma+s)} \leq A_{j\cdot}^{r+(t_1+t_2+1)\gamma}.
\end{aligned}
$$

Since $r \geq n^2$, by Proposition 4.4 $A_{\cdot i}^r = A_{\cdot i}^{r+(t_1+t_2+1)\gamma}$ and $A_{j\cdot}^r = A_{j\cdot}^{r+(t_1+t_2+1)\gamma}$, hence all inequalities (31) are equalities. Multiplying them by $a_{ij}^{(t_1\gamma+l)}$ we obtain (29), which is (30) in the visualized case. $\square$

Proposition 4.6 says that in any power A^r for $r \geq n^2$, the critical columns (or rows) can be obtained from the critical columns (or rows) of the spectral projector $Q(A^\gamma)$ via a permutation whose cycles are determined by the cyclic classes of $\mathcal{C}(A)$. Proposition 4.5 adds to this that all non-critical columns (or rows) of any periodic power are in the max cone spanned by the critical columns (or rows). From this we conclude the following.

Proposition 4.7. *All powers A^r for $r \geq T(A)$ have the same column span, which is the eigencone $V(A^\gamma)$.*

Proposition 4.7 enables us to say that $V(A^\gamma)$ is the *ultimate column span* of A. Similarly, we have the *ultimate row span* which is $V((A^T)^\gamma)$. These cones are generated by critical columns (or rows) of the Kleene star $(A^\gamma)^*$. For a basis of this cone, we can take any set of columns $(A^\gamma)^*$ (equivalently $Q(A^\gamma)$ or A^r for $r \geq T(A)$), whose indices form a minimal set of representatives of all cyclic classes of $\mathcal{C}(A)$. This basis is tropically independent in the sense of [2, 28, 27].

4.3. **Solving periodicity problems by square multiplication.** Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda(A) = 1$. The *t-attraction cone* $\text{Attr}(A, t)$ is the max cone which consists of all vectors x, for which there exists an integer r such that $A^r \otimes x = A^{r+t} \otimes x$, and hence this is also true for all integers greater than or equal to r. Actually we may speak of any $r \geq T(A)$, due to the following observation.

Proposition 4.8. *Let A be irreducible and definite. The systems $A^r \otimes x = A^{r+t} \otimes x$ are equivalent for all $r \geq T(A)$.*

Proof. Let x satisfy $A^s \otimes x = A^{s+t} \otimes x$ for some $s \geq T(A)$, then it also satisfies this system for all greater s. Due to the periodicity, for all k from $T(A) \leq k \leq s$ there exists $l > s$ such that $A^k = A^l$. Hence $A^k \otimes x = A^{k+t} \otimes x$ also hold for $T(A) \leq k \leq s$. $\square$

Corollary 4.9. $\text{Attr}(A, t) = \text{Attr}(A^t, 1)$.

Proof. By Proposition 4.8, $\text{Attr}(A, t)$ is solution set to the system $A^r \otimes x = A^{r+t} \otimes x$ for any $r \geq T(A)$ which is a multiple of t, which proves the statement. $\square$

An equation of $A^r \otimes x = A^{r+t} \otimes x$ with index in $\{1, \ldots, c\}$ will be called *critical*, and the subsystem of the first c equations will be called the *critical subsystem*.

Proposition 4.10. *Let A be irreducible and definite and let $r \geq T(A)$. Then $A^r \otimes x = A^{r+t} \otimes x$ is equivalent to its critical subsystem.*

Proof. Consider a non-critical component $A^r_{k.} \otimes x = A^{r+t}_{k.} \otimes x$. Using (24) it can be written as

$$(32) \qquad \bigoplus_{i=1}^c a^{(r)}_{ki} A^r_{i.} \otimes x = \bigoplus_{i=1}^c a^{(r)}_{ki} A^{r+t}_{i.} \otimes x,$$

hence it is a max combination of equations in the critical subsystem. $\qquad\square$

Next we give a bound on the computational complexity of deciding whether $x \in \text{Attr}(A, t)$, as well as other related problems which we formulate below.

P1. For a given x, decide whether $x \in \text{Attr}(A, t)$.

P2. For a given $k : 0 \leq k < \gamma$, compute periodic power A^r where $r \equiv k (\text{mod } \gamma)$.

P3. For a given x compute the ultimate period of $\{A^r \otimes x, \ r \geq 0\}$, meaning the least integer α such that $A^{r+\alpha} \otimes x = A^r \otimes x$ for all $r \geq T(A)$.

The following theorem is analogous to the results of Semančíková [39, 40].

Theorem 4.11. *For any irreducible matrix $A \in \mathbb{R}^{n \times n}_+$, the problems P1-P3 can be solved in $O(n^3 \log n)$ time.*

Proof. First note that we can compute both $\lambda(A)$ and a subeigenvector, and identify all critical nodes in no more than $O(n^3)$ operations, which is done essentially by Karp and Floyd-Warshall algorithms [35]. Further we can identify all cyclic classes of $\mathcal{C}(A)$ by Balcer-Veinott condensation in $O(n^2)$ operations.

By Proposition 4.4 the critical rows and columns become periodic for $r \geq n^2$. To know the critical rows and columns of a given power $r' \geq T(A)$, it suffices to compute A^r for arbitrary $r \geq n^2$ which can be done in $O(\log n)$ matrix squaring $(A, A^2, A^4, \ldots)$ and takes $O(n^3 \log n)$ time, and to apply the corresponding permutation on cyclic classes which takes $O(n^2)$ overrides. By Proposition 4.10 we readily solve P1 by the verification of the critical subsystem of $A^{r'} \otimes x = A^{r'+t} \otimes x$ which takes $O(n^2)$ operations. Using linear dependence (24) the remaining non-critical submatrix of A^r, for any $r \geq T(A)$ such that $r \equiv k (\text{mod } \gamma)$, can be computed in $O(n^3)$ time. This solves P2.

As the non-critical rows of A are generated by the critical rows, the ultimate period of $\{A^r \otimes x\}$ is determined by the critical components. For visualized matrix we know that $A^{r+t}_{i.} = A^r_{j.}$ for all i, j such that $[i] \to_t [j]$. This implies $(A^{r+t} \otimes x)_i = (A^r \otimes x)_j$ for $[i] \to_t [j]$, meaning that, to determine the period we need only the critical subvector of $A^r \otimes x$ for any

fixed $r \geq n^2$. Indeed, for any $i \in N_{\mathcal{C}}(A)$ and $r \geq n^2$ the sequence $\{(A^{r+t} \otimes x)_i, \ t \geq 0\}$ can be represented as a sequence of critical coordinates of $A^r \otimes x$ determined by a permutation on γ_μ cyclic classes of the s.c.c. to which i belongs. To compute the period, we take a sample of γ_μ numbers appearing consecutively in the sequence, and check all possible periods, which takes no more than γ_μ^2 operations. The period of $A^r \otimes x$ appears as the l.c.m. of these periods. It remains to note that all operations above do not require more than $O(n^3)$ time. This solves P3. $\square$

5. Properties of periodic powers

5.1. **Core matrix.** In the sequel we always assume that $A \in \mathbb{R}_+^{n \times n}$ is irreducible. Let $\mathcal{C}(A)$ consist of n_c s.c.c. $\mathcal{C}_\mu$ with cyclicities γ_μ, for $\mu = 1, \ldots, n_c$. Let $\gamma = \mathrm{l.c.m.}(\gamma_\mu)$ and $\overline{c}$ be the number of non-critical nodes. Further it will be convenient (though artificial) to consider, together with these components, also "non-critical components" $\mathcal{C}_\mu$ for $\mu = n_c + 1, \ldots, n_c + \overline{c}$, whose node sets N_μ consist of just one non-critical node, and the set of edges is empty.

Consider the block decomposition of A^r for $r \geq 1$, induced by the subsets N_μ for $\mu = 1, \ldots, n_c + \overline{c}$. The submatrix of A^r extracted form the rows in N_μ and columns in N_ν will be denoted by $A_{\mu\nu}^{(r)}$. If A is visualized and definite, we define the corresponding *core matrix* $A^{Core} = (\alpha_{\mu\nu})$, $\mu, \nu = 1, \ldots, n_c + \overline{c}$ by

(33) $$\alpha_{\mu\nu} = \max\{a_{ij} \mid i \in N_\mu, \ j \in N_\nu\}.$$

The entries of $(A^{Core})^*$ will be denoted by $\alpha_{\mu\nu}^*$. Their role is investigated in the next theorem.

Theorem 5.1. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite visualized matrix and $r \geq T_c(A)$. Let $\mu, \nu = 1, \ldots, n_c + \overline{c}$ be such that at least one of these indices is critical. Then the maximal entry of the block $A_{\mu\nu}^{(r)}$ is equal to $\alpha_{\mu\nu}^*$.*

Proof. The entry $\alpha_{\mu\nu}^*$ is the maximal weight over paths from μ to ν, with respect to the matrix A^{Core}. We take such a path $(\mu_1, \ldots, \mu_l)$ with maximal weight, where $\mu_1 := \mu$ and $\mu_l = \nu$. With this path we can associate a path π in $\mathcal{D}(A)$ defined by $\pi = \tau_1 \circ \sigma_1 \circ \tau_2 \circ \ldots \circ \sigma_{l-1} \circ \tau_l$, where τ_i are critical paths which entirely belong to the components $\mathcal{C}_{\mu_i}$, and σ_i are edges with maximal weight connecting $\mathcal{C}_{\mu_i}$ to $\mathcal{C}_{\mu_{i+1}}$. Such a path exists since any two nodes in the same component $\mathcal{C}_\mu$ can be connected to each other by critical paths if μ is critical, and if μ is non-critical then $\mathcal{C}_\mu$ consists just of one node. The weights of τ_i are equal to 1, hence the weight of π is equal to $\alpha_{\mu\nu}^*$. It follows from the definition of $\alpha_{\mu\nu}$ and $\alpha_{\mu\nu}^*$ that $\alpha_{\mu\nu}^*$ is the greatest weight over all paths which connect nodes in $\mathcal{C}_\mu$ to nodes in $\mathcal{C}_\nu$. As at least one of the indices μ, ν is critical, there is freedom in the choice of the paths τ_1 or τ_l which can be of arbitrary length. Assume w.l.o.g. that μ is critical.

Then for any r exceeding the length of $\sigma_1 \circ \tau_2 \circ \ldots \circ \sigma_{l-1} \circ \tau_l$ which we denote by $l_{\mu\nu}$, the block $A_{\mu\nu}^{(r)}$ contains an entry equal to $\alpha_{\mu\nu}^*$ which is the greatest entry of the block. Taking the maximum $T'(A)$ of $l_{\mu\nu}$ over all ordered pairs (μ, ν) with μ or ν critical, we obtain the claim for $r \geq T'(A)$. Evidently, $T'(A)$ can be replaced by $T_c(A)$. $\qquad\square$

5.2. CSR-representation.

For a definite visualized matrix $A \in \mathbb{R}_+^{n \times n}$, the statements of Propositions 4.5 and 4.6 can be combined in the following. Let $C \in \mathbb{R}_+^{n \times c}$ and $R \in \mathbb{R}_+^{c \times n}$ be matrices extracted from the first c columns (resp. rows) of $Q(A^\gamma)$ (or equivalently $(A^\gamma)^*$), and let $S := A^{[C]}$, the critical matrix of A defined by (17).

Theorem 5.2 (Schneider [37]). *Let $A \in \mathbb{R}_+^{n \times n}$ be definite and visualized. For $r \geq T(A)$ and $r \equiv l(\mathrm{mod}\ \gamma)$, $A^r = C \otimes S^l \otimes R$.*

Proof. By (21) $Q(A^\gamma) = C \otimes R$, and by Theorem 4.3 $A^{\gamma t} = Q(A^\gamma) = C \otimes R$ for $\gamma t \geq T(A)$. Thus the claim is already proved for $r = \gamma t \geq T(A)$.

Note that C (resp. R) can be extracted from the first c columns (resp. rows) of $A^{\gamma t}$.

As $S = A^{[C]}$, it follows that 1) the (i, j) entry of S^l can be 1 only if $[i] \to_l [j]$, 2) for each pair of classes $[i] \to_l [j]$ and each $i_1 \in [i]$ there exists $j_1 \in [j]$ such that the (i_1, j_1) entry of S^l equals 1.

Using these two observations and equation (29) applied to $A^{(\gamma t)}$ and $A^{(\gamma t + l)}$ for $\gamma t \geq T(A)$, we obtain that the critical columns of $A^{(\gamma t + l)}$ are given by $C \otimes S^l$ and the critical rows of $A^{(\gamma t + l)}$ are given by $S^l \otimes R$.

Combining this with any of the two equations of (24), we obtain that $A^{\gamma t + l} = C \otimes S^l \otimes R$, for any $\gamma t \geq T(A)$. $\qquad\square$

Further we observe that the dimensions of periodic powers and the *CSR-representation* established in Theorem 5.2 can be reduced.

The rows and columns with indices in the same cyclic class coincide in any power A^r, where $r \geq T_c(A)$ and A is definite and visualized. Hence the blocks of A^r, for $\mu, \nu = 1, \ldots, n_c + \bar{c}$ and $r \geq T_c(A)$, are of the form

$$
(34) \qquad A_{\mu\nu}^{(r)} = \begin{pmatrix} \tilde{a}_{s_1 t_1}^{(r)} E_{s_1 t_1} & \cdots & \tilde{a}_{s_1 t_m}^{(r)} E_{s_1 t_m} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{s_k t_1}^{(r)} E_{s_k t_1} & \cdots & \tilde{a}_{s_k t_m}^{(r)} E_{s_k t_m} \end{pmatrix},
$$

where k (resp. m) are cyclicities of $\mathcal{C}_\mu$ (resp. $\mathcal{C}_\nu$), indices $s_1, \ldots, s_k$ (resp. $t_1, \ldots t_m$) correspond to properly arranged cyclic classes of $\mathcal{C}_\mu$ (resp. $\mathcal{C}_\nu$), and $E_{s_i t_j}$ are matrices with appropriate dimensions with all entries equal to 1. We assume that $\mathcal{C}_\mu$ has just one "cyclic class" if μ is non-critical.

Formula (34) defines the matrix $\tilde{A}^{(r)} \in \mathbb{R}_+^{(\tilde{c}+\overline{c})\times(\tilde{c}+\overline{c})}$, where $\tilde{c}$ is the total number of cyclic classes, as matrix with entries $\tilde{a}_{s_i t_j}^{(r)}$. Corresponding to (34), this matrix has blocks

$$
(35) \qquad \tilde{A}_{\mu\nu}^{(r)} = \begin{pmatrix} \tilde{a}_{s_1 t_1}^{(r)} & \cdots & \tilde{a}_{s_1 t_m}^{(r)} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{s_k t_1}^{(r)} & \cdots & \tilde{a}_{s_k t_m}^{(r)} \end{pmatrix}.
$$

It follows that $\tilde{A}^{(r_1+r_2)} = \tilde{A}^{(r_1)} \otimes \tilde{A}^{(r_2)}$ for all $r_1, r_2 \geq T_c(A)$. In words, the multiplication of any two powers $A^{(r_1)}$ and $A^{(r_2)}$ for $r_1, r_2 \geq T_c(A)$ reduces to the multiplication of $\tilde{A}^{(r_1)}$ and $\tilde{A}^{(r_2)}$.

If we take $r = \gamma t + l \geq T(A)$ (instead of $T_c(A)$ above) and denote $\tilde{A} := \tilde{A}^{(\gamma t+1)}$, then due to the periodicity we obtain

$$
(36) \qquad \tilde{A}^{(\gamma t+l)} = \tilde{A}^{((\gamma t+1)l)} = \tilde{A}^l = \tilde{A}^{\gamma t+l},
$$

so that $\tilde{A}^{(r)}$ can be regarded as the rth power of $\tilde{A}$, for all $r \geq T(A)$.

Matrices C, R, and S^t for $t \geq (n-1)^2 + 1$ have the same block structure as in (34). This shows that the behavior of periodic powers A^r is fully described by

$$
(37) \qquad \tilde{A}^r = \tilde{C} \tilde{S}^l \tilde{R}, \quad r \equiv l(\mathrm{mod}\ \gamma),
$$

where $\tilde{S}$ is a $\tilde{c} \times \tilde{c}$ Boolean matrix with blocks

$$
(38) \qquad \tilde{S}_{\mu\mu} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{S}_{\mu\nu} = 0 \text{ for } \mu \neq \nu,
$$

and $\tilde{C} \in \mathbb{R}_+^{(\tilde{c}+\overline{c})\times\tilde{c}}$ and $\tilde{R} \in \mathbb{R}_+^{\tilde{c}\times(\tilde{c}+\overline{c})}$ are extracted from critical columns and rows of $\tilde{A}^\gamma = Q(\tilde{A}^\gamma)$, or equivalently, formed from the scalars in the blocks of C and R.

5.3. Circulant properties.

Matrix $A \in \mathbb{R}_+^{n \times n}$ is called a *circulant* if there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $a_{ij} = \alpha_d$ whenever $j - i = d(\mathrm{mod}\ n)$. This looks like

$$
(39) \qquad A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_n & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_n & \alpha_1 & \cdots & \alpha_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_2 & \alpha_3 & \cdots & \cdots & \alpha_1 \end{pmatrix}
$$

We also consider the following generalizations of this notion.

Matrix $A \in \mathbb{R}_+^{m \times n}$ will be called a *rectangular circulant*, if $a_{ij} = a_{ps}$ whenever $p = i + t(\text{mod } m)$ and $s = j + t(\text{mod } n)$, for all i, j, t. When $m = n$, this is an ordinary circulant given by (39).

Matrix $A \in \mathbb{R}_+^{m \times n}$ will be called a *block $k \times k$ circulant* when there exist scalars $\alpha_1, \ldots, \alpha_k$ and a block decomposition $A = (A_{ij})$, $i, j = 1, \ldots, k$ such that $A_{ij} = \alpha_d E_{ij}$ if $j - i = d(\text{mod } k)$, where all entries of blocks E_{ij} are equal to 1.

$A \in \mathbb{R}_+^{m \times n}$ is called *d-periodic* when $a_{ij} = a_{is}$ if $(s - j) \text{ mod } n$ is a multiple of d, and when $a_{ji} = a_{si}$ if $(s - j) \text{ mod } m$ is a multiple of d.

We give an example of 6×9 rectangular circulant A:

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \end{pmatrix}.$$

This example provides evidence that a rectangular $m \times n$ circulant consists of ordinary $d \times d$ circulant blocks where $d = \text{g.c.d.}(m, n)$. In particular, it is d-periodic. Also, there exist permutation matrices P and Q such that PAQ is a block circulant:

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

We formalize these observations in the following.

Proposition 5.3. *Let $A \in \mathbb{R}_+^{m \times n}$ be a rectangular circulant, and let $d = \text{g.c.d.}(m, n)$.*

1. *A is d-periodic.*
2. *There exist permutation matrices P and Q such that PAQ is a block $d \times d$ circulant.*

Proof. 1. There are integers t_1 and t_2 such that $d = t_1 m + t_2 n$. Using the definition of rectangular circulant we obtain $a_{ij} = a_{is}$, if $s = j + t_1 m(\text{mod } n)$, and hence if $s = j + d(\text{mod } n)$. Analogously for rows, we obtain that $a_{ji} = a_{si}$, if $s = j + t_2 n(\text{mod } m)$, and hence if $s = j + d(\text{mod } m)$.

2. As A is d-periodic, all rows such that $i + d = j(\text{mod } m)$ are equal, so that $\{1, \ldots, m\}$ can be divided in d groups each with m/d indices, in such a way that $A_{i.} = A_{j.}$ if i and j belong to the same group. We can find a permutation matrix P such that $A' = PA$ will have

rows $A'_{1\cdot} = \ldots = A'_{d\cdot} = A_{1\cdot}$, $A'_{d+1\cdot} = \ldots = A'_{2d\cdot} = A_{2\cdot}$, and so on. Analogously we can find a permutation matrix Q such that $A'' = PAQ$ will have columns $A''_{\cdot 1} = \ldots = A''_{\cdot d} = A'_{\cdot 1}$, $A''_{\cdot d+1} = \ldots = A''_{\cdot 2d} = A'_{\cdot 2}$, and so on. Then A'' has blocks (A''_{ij}) for $i, j = 1, \ldots, d$. of dimension $n/d \times m/d$, where $A''_{ij} = a_{ij} E_{ij}$, and E_{ij} has all entries 1. As A is d-periodic, it can be shown that its submatrix extracted from the first d rows and columns, is a circulant. Hence A'' is a block circulant. $\qquad\square$

When g.c.d.$(m, n) = 1$, the rectangular circulant is 1-periodic, i.e., a constant matrix.

Proposition 5.4. *Let $A \in \mathbb{R}_+^{n \times n}$ be a definite visualized matrix which admits block decomposition (34), and $r \geq T(A)$. Let $\mathcal{C}_\mu, \mathcal{C}_\nu$ be two (possibly equal) components of $C(A)$, and $d = g.c.d.(\gamma_\mu, \gamma_\nu)$.*
   1. *$\tilde{A}_{\mu\nu}^{(r)}$ is a rectangular circulant (which is a circulant if $\mu = \nu$).*
   2. *For any critical μ and ν, there is a permutation P such that $(P^T \tilde{A} P)_{\mu\nu}^{(r)}$ is a block $d \times d$ circulant matrix.*
   3. *If r is a multiple of γ, then $\tilde{A}_{\mu\mu}^{(r)}$ are circulant Kleene stars, where all off-diagonal entries are strictly less than 1.*

Proof. 1.: Using Eqn. (30) we see that for all (i, j) and (k, l) such that $k = i + t (\mathrm{mod}\, \gamma_\mu)$ and $l = j + t (\mathrm{mod}\, \gamma_\nu)$,
$$\tilde{a}_{s_k t_l}^{(r)} = \tilde{a}_{s_i t_l}^{(r+t)} = \tilde{a}_{s_i t_j}^{(r)}.$$

2.: If $\mu = \nu$ then $P = I$, and if $\mu \neq \nu$ then P is any permutation matrix such that its "subpermutations" for N_μ and N_ν are given by P and Q of Proposition 5.3.

3.: Part 1 shows that $\tilde{A}_{\mu\mu}^{(r)}$ are circulants for any $r \geq T(A)$ and critical μ. If r is a multiple of γ, then $\tilde{A}_{\mu\mu}^{(r)}$ are submatrices of $\tilde{A}^\gamma = Q(\tilde{A}^\gamma)$ and hence of $(\tilde{A}^\gamma)^*$. This implies, using (6), that they are Kleene stars. As the μth component of $\mathcal{C}(\tilde{A})$ is just a cycle of length γ_μ, the corresponding component of $\mathcal{C}(\tilde{A}^\gamma)$ consists of γ_μ loops, showing that the off-diagonal entries of $\tilde{A}_{\mu\mu}^{(r)}$ are strictly less than 1. $\qquad\square$

6. ATTRACTION CONES

6.1. **A system for attraction cone.** Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda(A) = 1$. Recall that *attraction cone* Attr(A, t), where $t \geq 1$ is an integer, is the set which consists of all vectors x, for which there exists an integer r such that $A^r \otimes x = A^{r+t} \otimes x$, and hence this is also true for all integers greater than or equal to r. We have shown above, see Subsect. 4.3, that Attr(A, t) is solution set to the critical subsystem of the system $A^{r+t} \otimes x = A^r \otimes x$, for any $r \geq T_c(A)$. Next we show how the specific circulant structure of A^r at $r \geq T_c(A)$ can be exploited, to derive a more concise system of equations for the attraction cone Attr$(A, 1)$. Due to Theorem 5.1 the core matrix $A^{Core} = \{\alpha_{\mu\nu} \mid \mu, \nu = 1, \ldots, n_c\}$, and its Kleene

star $(A^{Core})^* = \{\alpha_{\mu\nu}^* \mid \mu,\nu = 1,\ldots,n_c\}$ will be of special importance. We introduce the notation

(40)
$$M_\nu^{(r)}(i) = \{j \in N_\nu \mid a_{ij}^{(r)} = \alpha_{\mu\nu}^*\}, \ i \in N_\mu, \ \forall \nu : \mathcal{C}_\nu \neq \mathcal{C}_\mu,$$
$$K^{(r)}(i) = \{t > c \mid a_{it}^{(r)} = \alpha_{\mu\nu(t)}^*\}, \ i \in N_\mu,$$

where $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$ are s.c.c. of $\mathcal{C}(A)$, N_μ and N_ν are their node sets, and $\nu(t)$ in the second definition denotes the index of the non-critical component which consists of the node t. The sets $M_\nu^{(r)}(i)$ are non-empty for any $r \geq T_c(A)$, due to Theorem 5.1.

The results of Subsect. 5.3 lead to the following properties of $M_\nu^{(r)}(i)$ and $K^{(r)}(i)$.

Proposition 6.1. *Let $r \geq T_c(A)$ and $\mu,\nu \in \{1,\ldots,n_c\}$.*

1. *If $[i] \rightarrow_t [j]$ and $i,j \in N_\mu$ then $M_\nu^{(r+t)}(i) = M_\nu^{(r)}(j)$ and $K^{(r+t)}(i) = K^{(r)}(j)$.*
2. *Each $M_\nu^{(r)}(i)$ is the union of some cyclic classes of $\mathcal{C}_\nu$.*
3. *Let $i \in N_\mu$ and $d = g.c.d.(\gamma_\mu, \gamma_\nu)$. Then, if $[p] \subseteq M_\nu^{(r)}(i)$ and $[p] \rightarrow_d [s]$ then $[s] \subseteq M_\nu^{(r)}(i)$.*
4. *Let $i,j \in N_\mu$ and $p,s \in N_\nu$. Let $[i] \rightarrow_t [j]$ and $[p] \rightarrow_t [s]$. Then $[p] \subseteq M_\nu^{(r)}(i)$ if and only if $[s] \subseteq M_\nu^{(r)}(j)$.*

Next we establish the cancellation rules which will enable us to write out a concise system of equations for the attraction cone $\mathrm{Attr}(A,1)$.

If $a < c$, then

(41)
$$\{x \colon ax \oplus b = cx \oplus d\} \ = \ \{x \colon b = cx \oplus d\}.$$

Now consider a system of equations over max algebra:

(42)
$$\bigoplus_{i=1}^n a_{1i}x_i \oplus c_1 = \bigoplus_{i=1}^n a_{2i}x_i \oplus c_2 = \ldots = \bigoplus_{i=1}^n a_{ni}x_i \oplus c_n.$$

Suppose that $\alpha_1,\ldots,\alpha_n \in \mathbb{R}_+$ are such that $a_{li} \leq \alpha_i$ for all l and i, and $S_l = \{i \mid a_{li} = \alpha_i\}$ for $l = 1,\ldots,n$. Let S_l be such that $\bigcup_{l=1}^n S_l = \{1,\ldots,n\}$. Repeatedly applying the elementary cancellation law described above, we obtain that (42) is equivalent to

(43)
$$\bigoplus_{i \in S_1} \alpha_i x_i \oplus c_1 = \bigoplus_{i \in S_2} \alpha_i x_i \oplus c_2 = \ldots = \bigoplus_{i \in S_n} \alpha_i x_i \oplus c_n.$$

We will refer to the equivalence between (42) and (43), which we acknowledge to [18], as to *chain cancellation.*

Using notation (40) and Proposition 6.1 we formulate the main result of the paper.

Theorem 6.2. *Let $A \in \mathbb{R}_+^{n \times n}$ be a visualized matrix and $r \geq T(A)$ be a multiple of γ. Then the system $A^r \otimes x = A^{r+1} \otimes x$ is equivalent to*

(44)
$$\bigoplus_{k \in [i]} x_k \oplus \bigoplus_{\mathcal{C}_\nu \neq \mathcal{C}_\mu} \alpha_{\mu\nu}^* \left(\bigoplus_{k \in M_\nu^{(r)}(i)} x_k \right) \oplus \bigoplus_{t \in K^{(r)}(i)} \alpha_{\mu\nu(t)}^* x_t =$$

$$= \bigoplus_{k \in [j]} x_k \oplus \bigoplus_{\mathcal{C}_\nu \neq \mathcal{C}_\mu} \alpha_{\mu\nu}^* \left(\bigoplus_{k \in M_\nu^{(r)}(j)} x_k \right) \oplus \bigoplus_{t \in K^{(r)}(j)} \alpha_{\mu\nu(t)}^* x_t,$$

where $\mathcal{C}_\mu$ is the component of $\mathcal{C}(A)$ which contains both $[i]$ and $[j]$, and $[i]$ and $[j]$ range over all pairs of cyclic classes such that $[i] \to_1 [j]$.

Proof. By Proposition 4.10 $A^r \otimes x = A^{r+1} \otimes x$ is equivalent to its critical subsystem. Consider a critical component of $A^r \otimes x = A^{r+1} \otimes x$:

(45)
$$\bigoplus_k a_{ik}^{(r)} x_k = \bigoplus_k a_{ik}^{(r+1)} x_k, \ i = 1, \ldots, c.$$

Consider j such that $[i] \to_1 [j]$. Then by Proposition 4.6, $a_{ik}^{(r+1)} = a_{jk}^{(r)}$, hence the critical subsystem of $A^r \otimes x = A^{r+1} \otimes x$ is as follows:

(46)
$$\bigoplus_k a_{ik}^{(r)} x_k = \bigoplus_k a_{jk}^{(r)} x_k, \ \forall i, j : \ [i] \to_1 [j].$$

The number of non-identical equalities in (45) and (46) is equal to the total number of cyclic classes.

Proposition 5.4, part 3, implies that all principal submatrices of A^r extracted from critical components have a circulant block structure. In this structure, all entries of the diagonal blocks are equal to 1, and the entries of all off-diagonal blocks are strictly less than 1. Hence we can apply the chain cancellation (equivalence between (42) and (43)) and obtain the first terms on both sides of (44). By Theorem 5.1 each block $A_{\mu\nu}$ contains an entry equal to $\alpha_{\mu\nu}^*$. For a non-critical $\nu(t)$, this readily implies that the corresponding "subcolumn" $A_{\mu\nu(t)}$ contains an entry $\alpha_{\mu\nu(t)}^*$. Applying the chain cancellation we obtain the last terms on both sides of (44). Due to the block circulant structure of $A_{\mu\nu}$ with both μ and ν critical, see Proposition 5.4 or Proposition 6.1, we see that each column of such block also contains an entry equal to $\alpha_{\mu\nu}^*$. Applying the chain cancellation we obtain the remaining terms in (44). $\square$

As $\mathrm{Attr}(A, t) = \mathrm{Attr}(A^t, 1)$, system (44) also describes more general attraction cones, it only amounts to substitute $\mathcal{C}(A^t)$ for $\mathcal{C}(A)$ and the entries of $((A^t)^{Core})^*$ for $\alpha_{\mu\nu}^*$ (the dimension of this matrix will be different in general, see Proposition 3.3 part 3).

We note that the system for $\mathrm{Attr}(A, 1)$ naturally breaks into several chains of equations corresponding to the s.c.c. of $\mathcal{C}(A)$. If we start with (46), it can be equivalently written as

$R \otimes x = H \otimes y$ where R is the factor in CSR-representation, and $H \in \mathbb{R}_+^{c \times n_c}$ is a Boolean matrix with entries

$$(47) \qquad h_{i\mu} = \begin{cases} 1, & \text{if } i \in N_\mu, \\ 0, & \text{otherwise.} \end{cases}$$

We can apply cancellation as described in the proof of Theorem 6.2, to get rid of redundant terms on the left-hand side of the two-sided system.

If $\mathcal{C}(A)$ is strongly connected then H is a vector of all ones, and the two-sided system $R \otimes x = H \otimes y$ becomes essentially one-sided. We treat this case in the next subsections.

6.2. Extremals of attraction cones.

System (44) in general consists of several chains of equations corresponding to s.c.c. of $\mathcal{C}(A)$. Each chain is of the form

$$(48) \qquad \bigoplus_{i \in T_1} a_i x_i = \bigoplus_{i \in T_2} a_i x_i = \ldots = \bigoplus_{i \in T_m} a_i x_i,$$

where $T_1 \cup \ldots \cup T_m = \{1, \ldots, n\}$ and a_i come from the entries of $(A^{Core})^*$.

Here we consider only the case of strongly connected $\mathcal{C}(A)$, i.e., only one chain. By scaling $y_i = a_i x_i$ we obtain

$$(49) \qquad \bigoplus_{i \in T_1} y_i = \bigoplus_{i \in T_2} y_i = \ldots = \bigoplus_{i \in T_m} y_i,$$

Note that when the critical graph is not strongly connected, we have several chains of equations and the coefficients of (48) in general cannot be scaled to get (49) for each chain at the same time.

By e^i we denote the vector which has the ith coordinate and all the rest equal to 0. Vector $y \in \mathbb{R}_+^n$ will be called *scaled* if $\bigoplus_{i=1}^n y_i = 1$, and set S is called scaled if it consists of scaled vectors. We say that $V \subseteq \mathbb{R}_+^n$ is *generated* by $S \subseteq \mathbb{R}_+^n$ (also, S is a generating set of V) if $S \subseteq V$ and for any $x \in V$ there exist $y^1, \ldots, y^l \in S$ and $\alpha_1, \ldots, \alpha_l \in \mathbb{R}_+$ such that $x = \bigoplus_{i=1}^l \alpha_i y^i$.

We investigate extremal solutions of (49): a solution x is called *extremal* if $x = y \oplus z$ for two other solutions y, z implies that $x = y$ or $x = z$ [10]. The following can be deduced from the results of [9, 10].

Proposition 6.3. *The solution set of any finite system of max-linear equations has a finite generating set. In particular, it is generated by extremal solutions, and any set of scaled generators for the solution set contains all scaled extremal solutions.*

In the next proposition we show that extremal solutions of (49) can have only 0 and 1 components.

Proposition 6.4. *Let y be a scaled solution of* (49) *and let $0 < y_i < 1$ for some i. Then y is not an extremal.*

Proof. Let $K^< := \{i \mid 0 < y_i < 1\}$ and $K^1 := \{i \mid y_i = 1\}$, and define vectors v^0 and $v^1(k)$ for each $k \in K^<$ by

(50) $$v_i^0 = \begin{cases} 1, & \text{if } i \in K^1 \\ 0, & \text{otherwise} \end{cases}, \quad v_i^1(k) = \begin{cases} 1, & \text{if } i \in K^1 \cup \{k\} \\ 0, & \text{otherwise} \end{cases}.$$

Observe that both v^0 and $v^1(k)$ for any $k \in K^<$, are solutions to (49), different from y. We have:

(51) $$y = v^0 \oplus \bigoplus_{k \in K^<} y_k \cdot v^1(k),$$

hence y is not an extremal. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Let $T = (t_{ij})$ be the $m \times n$ $0 - 1$ matrix defined by

(52) $$t_{ij} = \begin{cases} 1, & \text{if } j \in T_i, \\ 0, & \text{otherwise}, \end{cases}$$

where T_i are from (49).

A subset $K \subseteq \{1, \ldots, n\}$ is called a *covering* of T if each T_i contains an index from K. The following is immediate.

Proposition 6.5. *A scaled vector y is a solution of* (49) *if and only if $K^1 := \{i \mid y_i = 1\}$ is a covering of T.*

A covering K is called *minimal* if it does not contain any proper subset which is also a covering.

A covering K will be called *nearly minimal* if it contains no more than one proper subcovering K'. Observe that then the complement $K \backslash K'$ consists of just one index. Hence, a covering is nearly minimal if and only if there may exist no more than one $i \in K$ such that $K \backslash \{i\}$ is a covering.

Proposition 6.6. *Extremal solutions of* (49) *are precisely $v^K = \bigoplus_{i \in K} e_i$, where K is a nearly minimal covering of T.*

Proof. If a covering K is not nearly minimal, then there exist i and j such that $K[i] := K \backslash \{i\}$ and $K[j] := K \backslash \{j\}$ are both coverings of T. Then $v^{K[i]}$ and $v^{K[j]}$ are both solutions and $v = v^{K[i]} \oplus v^{K[j]}$ hence v^K is not extremal.

Conversely, if v^K is not extremal, then there exist $y \neq v^K$ and $z \neq v^K$ such that $v^K = y \oplus z$. Evidently $y \leq v^K$ and $z \leq v^K$. By Proposition 6.4 we can represent y and z

as combinations of $0-1$ solutions of (49). These solutions correspond to coverings, which must be proper subsets of K. At least two of these coverings must be different from each other, hence K is not nearly minimal. $\qquad\square$

Thus, the problem of finding all nearly minimal coverings of a $0-1$ matrix is equivalent to the problem of finding all extremal solutions of (49).

The following case applies if the critical graph is strongly connected and occupies all nodes.

Corollary 6.7. *Let $T_1, \ldots, T_m$ be pairwise disjoint, then the scaled extremals are precisely all vectors $v^S = \bigoplus_{i \in S} e^i$, where S is an index set which contains exactly one index from each set T_i.*

Proof. Any such set S forms a minimal covering of T, and it can be shown that the solution set of (49) is generated by v^S, so there are no more scaled extremals (or nearly minimal coverings). $\qquad\square$

6.3. **An algorithm for finding the coefficients of an attraction system.** Coefficients of the system of equations which defines attraction cone are determined by the entries of $(A^{Core})^*$ which can be found in $O((n_c + \overline{c})^3)$ operations, where n_c is the number of s.c.c. of $\mathcal{C}(A)$ and $\overline{c}$ is the number of non-critical nodes. However it remains to find the places where these coefficients appear, i.e., the sets $M_\nu^{(r)}(i)$ and $K^{(r)}(i)$ for $i = 1, \ldots, c$. Solving this problem, we get another polynomial method for computing the coefficients of (44).

Here we restrict our attention to the case when $\mathcal{C}(A)$ is strongly connected. In this case there are no second terms on both sides of (44) and we need only $K^{(r)}(i)$. The digraph $\mathcal{D}(A^{Core})$ associated with the matrix A^{Core} consists of one critical node which corresponds to the whole $\mathcal{C}(A)$ and will be denoted by μ, and $\overline{c}$ non-critical nodes $\nu(t)$, for $t > c$. The core matrix is of the form

$$(53) \qquad A^{Core} = \begin{pmatrix} 1 & h \\ g & B \end{pmatrix},$$

the entries $\alpha_{\mu\nu}$ and the entries of h, g and $B = (b_{\nu(s),\nu(t)})$ are given by

$$\alpha_{\mu\mu} = 1,$$
$$(54) \qquad h_{\nu(t)} = \alpha_{\mu\nu(t)} = \max_{k=1}^{c} a_{kt}, \ \ g_{\nu(t)} = \alpha_{\nu(t)\mu} = \max_{k=1}^{c} a_{tk}, \ \ t > c,$$
$$b_{\nu(s)\nu(t)} = \alpha_{\nu(s)\nu(t)} = a_{st}, \ \ s > c, \ t > c.$$

Denote by $[\to_m i]$ the cyclic class $[j]$ such that $[j] \to_m [i]$. For each $t = 1, \ldots, c$, we initialize Boolean c-vectors P_t by

$$(55) \qquad P_t(i) = \begin{cases} 1, & \text{if } [\to_1 i] \cap \arg \max_{k=1}^c a_{kt} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

P_t encode the Boolean information associated with h (shifted cyclically by 1).

Further we compute the Kleene star of the non-critical submatrix $B := A_{MM}$, where M denotes the set of non-critical nodes, and store the information on the lengths of paths with maximal weight and length not exceeding $\bar{c}$ in sets U_{st} associated to each entry of B. To compute these sets we use the formula

$$(56) \qquad B^* = I \oplus B \oplus \ldots \oplus B^{\bar{c}-1},$$

where for each entry of this matrix series we find the arguments of maxima.

To combine the information associated with h and B^*, we recall the max-algebraic version of bordering method [11], which computes

$$(57) \qquad (A^C)^* = \begin{pmatrix} 1 & h^T \\ g & B \end{pmatrix}^* = \begin{pmatrix} 1 & h^T \otimes B^* \\ B^* \otimes g & B^* \oplus B^* \otimes g \otimes h^T \otimes B^* \end{pmatrix},$$

where $h, g \in \mathbb{R}_+^{\bar{c}}$. Note that all information that we need for system (44), is in the entries of $h^T \otimes B^*$ and in the indices of equations of the system where the entries of $h^T \otimes B^*$ appear. Computing $(h^T \otimes B^*)_i$ means in particular obtaining the "winning" indices

$$(58) \qquad W_t = \arg \max_{s > c} h_{\nu(s)} b^*_{\nu(s)\nu(t)}.$$

After that, the idea is to combine P_s with U_{st} for all $s \in W_t$ and unite the obtained indices. More precisely, for each number m stored in U_{st} we define the shifted Boolean vector $P_s^{\to m}$ by

$$(59) \qquad P_s^{\to m}(i) = \begin{cases} 1, & \text{if } [\to_{m+1} i] \cap \arg \max_{k=1}^c a_{ks} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$(60) \qquad P_s^{\to m}(i) = 1 \Leftrightarrow P_s(j) = 1 \text{ and } [j] \to_m [i].$$

After that, we define

$$(61) \qquad G_t := \bigvee_{s \in W_t} \bigvee_{m \in U_{st}} P_s^{\to m}.$$

Proposition 6.8. *Let $t > c$, and let $r \geq T(A)$ be a multiple of γ. Then for all $i \leq c$, $t \in K^{(r)}(i)$ if and only if $G_t(i) = 1$.*

Proof. $G_t(i) = 1$ if and only if there exist $s \in W_t$ and $m \in U_{st}$ such that $P_s^{\to m}(i) = 1$. Then there exists a path $\pi_1 \circ \tau \circ \pi_2$ where π_1 starts in i, belongs to $\mathcal{C}(A)$ and has length $-m-1 (\mathrm{mod}\ \gamma)$, τ is an edge which attains $h_{\nu(s)} = \max_{l=1}^c a_{ls}$, and π_2 is entirely non-critical, has length m, weight $b^*_{\nu(s)\nu(t)}$, and connects s to t. The weight of $\pi_1 \circ \tau \circ \pi_2$ is equal to $1 \cdot h_s \cdot b^*_{st} = \alpha^*_{\mu\nu(t)}$, and the length is a multiple of γ, meaning that $a_{it}^{(r)} = \alpha^*_{\mu\nu(t)}$ and $t \in K^{(r)}(i)$.

The other way around, let $t \in K^{(r)}(i)$. Then there exists a path P of length r that connects i to t and has weight $\alpha^*_{\mu\nu(t)} = \bigoplus_s h_{\nu(s)} b^*_{\nu(s)\nu(t)}$. We can decompose $P = \pi'_1 \circ \tau' \circ \pi'_2$, where τ' is an edge connecting a critical node to a non-critical node, and π'_2 has only non-critical nodes. Obviously $w(\tau' \circ \pi'_2) \le \bigoplus_s h_{\nu(s)} b^*_{\nu(s)\nu(t)} = \alpha^*_{\mu\nu(t)}$. But $w(P) = \alpha^*_{\mu\nu(t)}$, hence $w(\pi'_1) = 1$ so that π'_1 entirely belongs to $\mathcal{C}(A)$, and $w(\tau' \circ \pi'_2) = \alpha^*_{\mu\nu(t)}$. In particular, $w(\tau') = h_{\nu(s)}$ and $w(\pi'_2) = b^*_{\nu(s)\nu(t)}$ for certain $s > c$. From this we conclude that $G_t(i) = 1$. $\qquad\square$

Summarizing above said, we have the following algorithm for computing the coefficients of (44) in the case when $\mathcal{C}(A)$ is strongly connected. Recall that in this case there is no second term on both sides of (44). The computation of coefficients of the third term includes the computation of $h \otimes B^*$ and the sets $K^{(r)}(i)$ for $i \le c$ (in fact we can improve the algorithm since only γ of them are different).

ALGORITHM 1. *Compute the coefficients of* (44) *if* $\mathcal{C}(A)$ *is strongly connected.*

Input. Visualized matrix A, critical graph $\mathcal{C}(A)$ which is strongly connected and the cyclic classes of $\mathcal{C}(A)$.

1. Compute h and initialize P_t for $t > c$. This takes $c\bar{c}$ operations.

2. Compute B^* and sets U_{st} for all $s, t > c$. It takes $O(\bar{c}^4)$ operations (see (56)).

3. Compute hB^* and G_t for $t > c$, by (58), (60) and (61). Computation of hB^* and W_t by (58) requires $O(\bar{c}^2)$ operations, computation of shifted Boolean vectors is $O(c\bar{c}^2)$, and the conjunction (61) takes $O(c\bar{c}^3)$ operations.

4. Compute $K^{(r)}(i)$ using Proposition 6.8. This requires $c\bar{c}$ operations.

As the overall complexity does not exceed $O(\bar{c}^4) + O(c\bar{c}^3) = O(n\bar{c}^3)$ operations, we conclude the following.

Proposition 6.9. *Let* $A \in \mathbb{R}_+^{n \times n}$ *be visualized,* $\mathcal{C}(A)$ *be strongly connected,* $\bar{c}$ *be the number of non-critical nodes, and suppose we know* $\mathcal{C}(A)$ *and all cyclic classes. Then Algorithm 1 computes the coefficients of the attraction system in no more than* $O(\bar{c}^3 n)$ *operations.*

It is also important that the eigenvalue and an eigenvector of irreducible matrix can be computed by the policy iteration algorithm of [12], which is very fast in practice. After

that, $\mathcal{C}(A)$ and the cyclic classes can be computed in $O(n^2)$ time. Thus we are led to an efficient method of computing the coefficients in the case when A is irreducible and $\mathcal{C}(A)$ is strongly connected, especially in the case when the number of non-critical nodes is small. Note that the case of irreducible A and strongly connected $\mathcal{C}(A)$ is generic when matrices A are real and generated at random. Also, in this generic case it almost never happens that maxima in blocks or among the weights of paths are achieved twice, which means that we do not need to assign Boolean vectors or sets to each entry. In this case the total number of operations in the algorithm is reduced to $O(\overline{c}^3) + O(\overline{c}c)$.

When $\mathcal{C}(A)$ is not strongly connected, the bordering method (57) can be used to obtain an algorithm which operates only with the entries of A^{Core}. However, the complexity of operations with indices and Boolean numbers significantly increases in that general case.

7. EXAMPLES

7.1. **Square multiplication.** In this subsection we will examine the problems that can be solved by matrix squaring on 9×9 real matrix over the *max-plus semiring*:

$$A = \begin{pmatrix} -1 & 0 & -1 & -1 & -9 & -7 & -10 & -4 & -8 \\ 0 & -1 & 0 & -1 & -10 & -1 & -10 & -9 & -4 \\ -1 & -1 & -1 & 0 & -2 & -3 & -2 & -6 & -6 \\ 0 & -1 & -1 & -1 & -10 & -6 & -10 & -6 & -1 \\ -10 & -2 & -8 & -1 & -1 & 0 & -1 & -10 & -1 \\ -5 & -5 & -10 & -9 & -1 & -1 & 0 & -3 & -6 \\ -9 & -10 & -7 & -10 & 0 & -1 & -1 & -8 & -8 \\ -75 & -80 & -77 & -83 & -80 & -77 & -82 & -2 & -0.5 \\ -84 & -81 & -77 & -80 & -78 & -77 & -78 & -0.5 & -2 \end{pmatrix}$$

The corresponding max-times example is obtained by, e.g., taking *exponents* of the entries.

The critical graph of A, see Figure 3, has two s.c.c.: C_1 with nodes $N_1 = \{1, 2, 3, 4\}$ and C_2 with nodes $N_2 = \{5, 6, 7\}$. The cyclicity of C_1 is $\gamma_1 = 2$ and the cyclicity of C_2 is $\gamma_2 = 3$, so the cyclicity of $\mathcal{C}(A)$ is $\gamma = \mathrm{lcm}(2, 3) = 2 \times 3 = 6$.
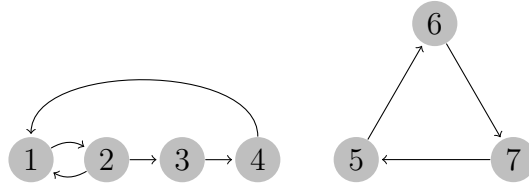

FIGURE 3. The critical graph of A

The matrix can be decomposed into blocks

$$
A = \begin{pmatrix} A_{11} & A_{12} & A_{1M} \\ A_{21} & A_{22} & A_{2M} \\ A_{M1} & A_{M1} & A_{MM} \end{pmatrix},
$$

where the submatrices A_{11} and A_{22} correspond to two s.c.c. C_1 and C_2 of $\mathcal{C}(A)$, see Figure 3. They equal

$$
A_{11} = \begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix},
$$

and A_{MM} is the non-critical principal submatrix

$$
A_{MM} = \begin{pmatrix} -2 & -0.5 \\ -0.5 & -2 \end{pmatrix}.
$$

The submatrices A_{12}, A_{21}, A_{1M} and A_{2M} are composed of randomly taken numbers from -1 to -10, and A_{M1} and A_{M2} are composed of randomly taken numbers from -75 to -85.

It can be checked that the powers of A become periodic after $T(A) = 154$.

We will consider the following instances of problems P2 and P3.

P2. Compute A^r for $r \geq T(A)$ and $r \equiv 2 \pmod 6$.

P3. For given $x \in \mathbb{R}_+^9$, find ultimate orbit period of $A^k \otimes x$.

Solving P2. Using the idea of Proposition 4.11, we perform 7 squarings $A, A^2, A^4, \ldots$ to raise A to the power $128 > 9 \times 9$. This brings us to the matrix

$$
A^{128} = \begin{pmatrix} A_{11}^{(128)} & A_{12}^{(128)} & A_{1M}^{(128)} \\ A_{21}^{(128)} & A_{22}^{(128)} & A_{2M}^{(128)} \\ A_{M1}^{(128)} & A_{M2}^{(128)} & A_{MM}^{(128)} \end{pmatrix},
$$

where

$$
A_{11}^{(128)} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad A_{22}^{(128)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix},
$$

all entries of $A_{12}^{(128)}$ and $A_{21}^{(128)}$ are -1 and

$$A_{1M}^{(128)} = \begin{pmatrix} -2.5 & -1 \\ -1.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix}, \quad A_{2M}^{(128)} = \begin{pmatrix} -1.5 & -2 \\ -2.5 & -2 \\ -2.5 & -1 \end{pmatrix}$$

$$A_{M1}^{(128)} = \begin{pmatrix} -76 & -75.5 \\ -75 & -76.5 \\ -76 & -75.5 \\ -75 & -76.5 \end{pmatrix}^T, \quad A_{M2}^{(128)} = \begin{pmatrix} -76 & -76.5 \\ -76 & -76.5 \\ -76 & -76.5 \end{pmatrix}^T$$

We are lucky since $128 \equiv 2(\text{mod } 6)$, as we already have true critical columns and rows of A^r. However, the non-critical principal submatrix of A^{128} is

$$A_{MM}^{(128)} = \begin{pmatrix} -64 & -65.5 \\ -65.5 & -64 \end{pmatrix}.$$

It can be checked that this is *not* the non-critical submatrix of A^r that we seek (recall that $T(A) = 154$). Hence, it remains to compute the principal non-critical submatrix $A_{MM}^{(r)}$.

We note that A^{132} has critical rows and columns of the spectral projector $Q(A)$, since 132 is a multiple of $\gamma = 6$. In A^{132}, the critical rows and columns $1 - 4$ (in C_1) are the same as that of A^{128}, since $\gamma_1 = 2$ and both 128 and 132 are even. The critical rows $5 - 7$ (in C_2) can be computed from those of A^{128} by cyclic permutation $(5, 6, 7) \to (7, 5, 6)$, and the critical rows $5 - 7$ can be computed by the inverse permutation $(5, 6, 7) \to (6, 7, 5)$. This implies that all blocks in A^{132} are the same as in A^{128} above (in the analogous block decomposition of A^{132}), except for

$$A_{22}^{(132)} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad A_{2M}^{(132)} = \begin{pmatrix} -2.5 & -2 \\ -2.5 & -1 \\ -1.5 & -2 \end{pmatrix}.$$

Now the remaining non-critical submatrix of A^r can be computed using linear dependence (24), which specifies to

$$A_{\cdot k}^{(r)} = \bigoplus_{i=1}^{7} a_{ik}^{(132)} A_{\cdot i}^{(128)}, \quad k = 8, 9.$$

This yields

$$A_{MM}^{(r)} = \begin{pmatrix} -76.5 & -77 \\ -78 & -76.5 \end{pmatrix}$$

Solving P3 We examine the orbit period of $A^k x$ for $x = x^1, x^2, x^3, x^4$, where

$$x^1 = [1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9],$$

$$x^2 = [1 \quad 2 \quad 3 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$x^3 = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1],$$

$$x^4 = [0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

We compute $y = A^{128} x$ for $x = x^1, x^2, x^3, x^4$:

$$y^1 = A^{128} \otimes x^1 = [8 \quad 7 \quad 8 \quad 7 \quad 7 \quad 7 \quad 8 \quad \times \quad \times],$$

$$y^2 = A^{128} \otimes x^2 = [3 \quad 4 \quad 3 \quad 4 \quad 3 \quad 3 \quad 3 \quad \times \quad \times],$$

$$y^3 = A^{128} \otimes x^3 = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad \times \quad \times],$$

$$y^4 = A^{128} \otimes x^4 = [1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \times \quad \times].$$

Here $\times$ correspond to non-critical entries which we do not need. The cyclic classes of C_1 are $\{1,3\}$, $\{2,4\}$, and the cyclic classes of C_2 are $\{5\}$, $\{6\}$ and $\{7\}$. From the considerations of Proposition 4.11, it follows that the coordinate sequences $\{(A^r x)_i, \ r \geq T(A)\}$ are

$$y_1, \ y_2, \ y_1, \ y_2, \ldots, \ \text{for } i = 1, 2, 3, 4,$$

$$y_5, \ y_6, \ y_7, \ y_5, \ y_6, \ y_7, \ldots, \ \text{for } i = 5, 6, 7.$$

Looking at $y^1, \ldots, y^4$ above, we conclude that the orbit of x^1 is of the largest possible period 6, the orbit of x^2 is of the period 2 (in other words, $x^2 \in \text{Attr}(A, 2)$), the orbit of x^3 is of the period 3 (i.e., $x^3 \in \text{Attr}(A, 3)$), and the orbit of x^4 is of the period 1 (i.e., $x^1 \in \text{Attr}(A, 1)$).

7.2. Circulants.

Here we consider a 9×9 example of definite and visualized matrix in *max-plus algebra*

$$(62) \qquad A = \begin{pmatrix}
-8 & 0 & -1 & -8 & -8 & -9 & -4 & -5 & -1 \\
-4 & -5 & 0 & -2 & -6 & 0 & -7 & -3 & -9 \\
-7 & -9 & -8 & 0 & -8 & -4 & -6 & -9 & -10 \\
-8 & -8 & -10 & -7 & 0 & -4 & -6 & -10 & -1 \\
-2 & -8 & -7 & -4 & -8 & 0 & -3 & -1 & -10 \\
0 & -1 & -2 & -7 & -10 & -6 & -3 & -6 & -1 \\
-10 & -7 & -7 & -7 & -6 & -1 & -5 & 0 & -9 \\
-8 & -3 & -6 & -8 & -6 & -8 & -5 & -10 & 0 \\
-4 & -3 & -5 & -6 & -6 & -10 & 0 & -6 & -9
\end{pmatrix}$$

The critical graph of this matrix consists of two s.c.c. comprising 6 and 3 nodes respectively. They are shown in Figures 4 and 5, together with their cyclic classes.
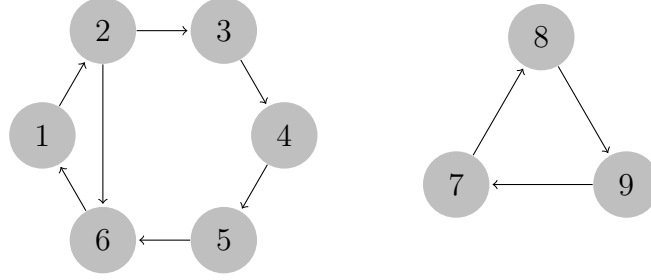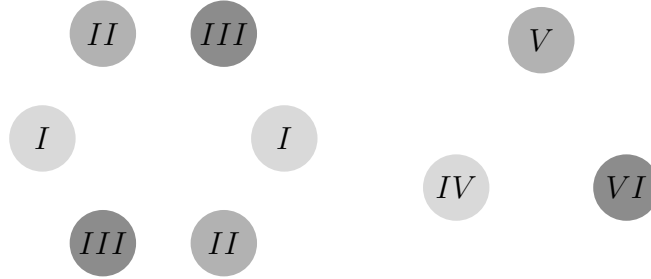

FIGURE 4. Critical graph of (62)

FIGURE 5. Cyclic classes of the critical graph

The components of $C(A)$ induce block decomposition

$$(63) \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$(64) \qquad A_{11} = \begin{pmatrix} -8 & 0 & -1 & -8 & -8 & -9 \\ -4 & -5 & 0 & -2 & -6 & 0 \\ -7 & -9 & -8 & 0 & -8 & -4 \\ -8 & -8 & -10 & -7 & 0 & -4 \\ -2 & -8 & -7 & -4 & -8 & 0 \\ 0 & -1 & -2 & -7 & -10 & -6 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} -5 & 0 & -9 \\ -5 & -10 & 0 \\ 0 & -6 & -9 \end{pmatrix}$$

The core matrix and its Kleene star are equal to

$$(65) \qquad A^{Core} = (A^{Core})^* = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

By calculating $A, A^2, \ldots$ we obtain that the powers of A become periodic after $T(A) = 6$. In the block decomposition of A^6 analogous to (63), we have the following circulants:

(66)
$$A^{(6)}_{11} = \begin{pmatrix} 0 & -1 & -2 & 0 & -1 & -2 \\ -2 & 0 & -1 & -2 & 0 & -1 \\ -1 & -2 & 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 & -1 & -2 \\ -2 & 0 & -1 & -2 & 0 & -1 \\ -1 & -2 & 0 & -1 & -2 & 0 \end{pmatrix}, \quad A^{(6)}_{12} = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \\ -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix},$$

$$A^{(6)}_{21} = \begin{pmatrix} -3 & -1 & -2 & -3 & -1 & -2 \\ -2 & -3 & -1 & -2 & -3 & -1 \\ -1 & -2 & -3 & -1 & -2 & -3 \end{pmatrix}, \quad A^{(6)}_{22} = \begin{pmatrix} 0 & -3 & -2 \\ -2 & 0 & -3 \\ -3 & -2 & 0 \end{pmatrix}.$$

The corresponding blocks of "reduced" power $\tilde{A}^{(6)}$ are

(67)
$$\tilde{A}^{(6)}_{11} = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}, \quad \tilde{A}^{(6)}_{12} = \begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix},$$

$$\tilde{A}^{(6)}_{21} = \begin{pmatrix} -3 & -1 & -2 \\ -2 & -3 & -1 \\ -1 & -2 & -3 \end{pmatrix}, \quad \tilde{A}^{(6)}_{22} = \begin{pmatrix} 0 & -3 & -2 \\ -2 & 0 & -3 \\ -3 & -2 & 0 \end{pmatrix}.$$

Note that $\tilde{A}^{(6)}_{11}$ and $\tilde{A}^{(6)}_{22}$ are Kleene stars, with all off-diagonal entries negative.

Using (66), we specialize system (44) to our case, we see that this system of equations for the attraction cone $\mathrm{Attr}(A, 1)$ consists of two chains of equations, namely

(68)
$$x_1 \oplus x_4 \oplus (x_8 - 1) \oplus (x_9 - 1) =$$
$$= x_2 \oplus x_5 \oplus (x_7 - 1) \oplus (x_9 - 1) = x_3 \oplus x_6 \oplus (x_7 - 1) \oplus (x_8 - 1),$$
$$(x_2 - 1) \oplus (x_5 - 1) \oplus x_7 =$$
$$= (x_3 - 1) \oplus (x_6 - 1) \oplus x_8 = (x_1 - 1) \oplus (x_4 - 1) \oplus x_9.$$

Note that only 0 and -1, the coefficients of $(A^{Core})^*$ (which is equal to A^{Core} in our example), appear in this system.

7.3. **Algorithm for the strongly connected case.** Here we consider a 6×6 max-plus example

(69)
$$
A = \begin{pmatrix}
-3 & 0 & -1 & -19 & -7 & -3 \\
-3 & -4 & 0 & -10 & -19 & -16 \\
0 & -3 & -1 & -10 & -8 & -8 \\
-1 & -4 & -4 & -1 & -1 & -3 \\
-1 & -1 & -4 & -2 & -4 & -1 \\
-4 & -2 & -4 & -1 & -4 & -1
\end{pmatrix},
$$

and apply to it the algorithm described in Subsect. 6.3. The critical graph of this matrix consists just of one cycle of length 3, and there are 3 non-critical nodes.

FIGURE 6. Critical graph and non-critical nodes of (69)

The core matrix in this case is equal to

$$
A^{Core} = \begin{pmatrix}
0 & -10 & -7 & -3 \\
-1 & -1 & -1 & -3 \\
-1 & -2 & -4 & -1 \\
-2 & -1 & -4 & -1
\end{pmatrix}
$$

Vector $h = (-10 \ -7 \ -3)^T$, whose components are computed by

(70)
$$
h_i = \bigoplus_{k=1}^{3} a_{ki}, \text{ for } i = 4, 5, 6,
$$

comprises $2, 3, 4$-components of the first row of A^{Core}. The arguments of maxima in (70) give, after the cyclic shift by one position, the Boolean vectors

(71)
$$
P_4 = (1 \ 0 \ 1), \ P_5 = (0 \ 1 \ 0), \ P_6 = (0 \ 1 \ 0).
$$

These vectors encode, for the corresponding non-critical nodes $t = 4, 5, 6$, the starting cyclic classes (here, just critical nodes!) of paths which go from $\mathcal{C}(A)$ directly to t and whose length is 3.

The non-critical principal submatrix of A and its Kleene star are equal to

$$B = \begin{pmatrix} -1 & -1 & -3 \\ -2 & -4 & -1 \\ -1 & -4 & -1 \end{pmatrix}, \quad B^* = \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix}$$

The lengths of optimal non-critical paths (whose weights are entries of B^*) can be written in the matrix of sets

(72)
$$U = \begin{pmatrix} \{0\} & \{1\} & \{2\} \\ \{1,2\} & \{0\} & \{1\} \\ \{1\} & \{2\} & \{0\} \end{pmatrix}$$

Further we compute

$$h^T \otimes B^* = (-10 \ -7 \ -3) \otimes \begin{pmatrix} 0 & -1 & -2 \\ -2 & 0 & -1 \\ -1 & -2 & 0 \end{pmatrix} = (-4 \ -5 \ -3)$$

The maxima in $\bigoplus_t h_t b_{ti}^*$ for all i are achieved only at $t = 6$, so $W_4 = W_5 = W_6 = \{6\}$. Hence G_4, G_5 and G_6 are shifted P_6 and the shift is determined by the components in the last row of U which is $(\{1\} \ \{2\} \ \{0\})$. From $P_6 = (0 \ 1 \ 0)$ we conclude that

$$G_4 = (0 \ 0 \ 1), \ G_5 = (1 \ 0 \ 0), \ G_6 = (0 \ 1 \ 0).$$

By Proposition 6.8 we have that $G_i(t) = 1$ if and only if $t \in K^{(r)}(i)$ (where $r \geq T(A)$ is a multiple of $\gamma = 3$). Using this rule we obtain that $K^{(r)}(1) = \{5\}$, $K^{(r)}(2) = \{6\}$, $K^{(r)}(3) = \{4\}$, and using the vector of coefficients $h^T \otimes B^* = (-4 \ -5 \ -3)$, we can write out the system for attraction cone

(73)
$$x_1 \oplus (x_5 - 5) = x_2 \oplus (x_6 - 3) = x_3 \oplus (x_4 - 4).$$

To verify this result, we observe that in our case $T(A) = 8$ and

$$
A^8 = \begin{pmatrix}
-1 & -1 & 0 & -4 & -6 & -4 \\
0 & -1 & -1 & -5 & -5 & -4 \\
-1 & 0 & -1 & -5 & -6 & -3 \\
-2 & -1 & -2 & -6 & -1 & -4 \\
-2 & -1 & -1 & -5 & -7 & -4 \\
-2 & -3 & -2 & -6 & -7 & -6
\end{pmatrix}
$$

$$
A^9 = \begin{pmatrix}
0 & -1 & -1 & -5 & -5 & -4 \\
-1 & 0 & -1 & -5 & -6 & -3 \\
-1 & -1 & 0 & -4 & -6 & -4 \\
-2 & -2 & -1 & -5 & -7 & -5 \\
-1 & -2 & -1 & -5 & -6 & -5 \\
-2 & -2 & -3 & -7 & -7 & -5
\end{pmatrix}
$$

Applying cancellation to the critical subsystem of $A^8 \otimes x = A^9 \otimes x$, we obtain (73).

8. Acknowledgement

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